m-cluster categories and m-replicated algebras

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Abstract

Let A be a hereditary algebra over an algebraically closed field. We prove that an exact fundamental domain for the m-cluster category $\mathcal{C}_m(A)$ of A is the m-left part $\mathcal{L}_m(A^{(m)})$ of the m-replicated algebra of A. Moreover, we obtain a one-to-one correspondence between the tilting objects in $\mathcal{C}_m(A)$ (that is, the m-clusters) and those tilting modules in mod $A^{(m)}$ for which all non projective-injective direct summands lie in $\mathcal{L}_m(A^{(m)})$.

Furthermore, we study the module category of $A^{(m)}$ and show that a basic exceptional module with the correct number of non-isomorphic indecomposable summands is actually a tilting module. We also show how to determine the projective dimension of an indecomposable $A^{(m)}$-module from its position in the Auslander-Reiten quiver.

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[1] Partially supported by the NSERC of Canada

[2] Partially supported by the NSERC of Canada and the universities of Sherbrooke

0 Introduction

Cluster categories were introduced in [11] and, for type A_n, also in [12] in order to understand better the cluster algebras of Fomin and Zelevinsky [15,16]. They are defined as follows. Let A be a hereditary algebra over an algebraically closed field, and $\mathcal{D}^b(\operatorname{mod} A)$ be the derived category of bounded complexes of finitely generated A-modules, then the cluster category is the orbit category of $\mathcal{D}^b(\operatorname{mod} A)$ under the action of the functor $F = \tau^{-1}[1]$, where τ is the Auslander-Reiten translation in $\mathcal{D}^b(\operatorname{mod} A)$ and $[1]$ is the shift. Later, the m-cluster category $\mathcal{C}_m(A)$ was introduced in [25] (see also [22,28,8]) as a means for encoding the combinatorics of m-clusters of Fomin and Reading [14] in a fashion similar to the way the cluster category encodes the combinatorics of clusters. It is defined to be the orbit category of $\mathcal{D}^b(\operatorname{mod} A)$ under the action of the functor $\tau^{-1}[m]$. By [21], this category is triangulated. It is proven in [25] that there exists a bijection between m-clusters and m-tilting sets in $\mathcal{C}_m(A)$, that is, maximal sets of indecomposables S such that $\operatorname{Ext}^i_{\mathcal{C}_m(A)}(X,Y) = 0$ for all X, Y in S and all i with $1 \leq i \leq m$ (then the object $T = \oplus_{X \in S} X$ is called a *tilting object in* $\mathcal{C}_m(A)$).

In [3], we have given an interpretation of the cluster category and its tilting objects in terms of modules over a finite dimensional algebra, namely the duplicated algebra of the original hereditary algebra A. Our objective in the present paper is to extend this characterization to the m-cluster category and its tilting objects.

Following [4], we define the m-replicated algebra of A to be the (finite dimensional) matrix algebra

$$A^{(m)} = \begin{bmatrix} A_0 & 0 & \dots & \dots & \dots & 0 \\ Q_1 & A_1 & 0 & \dots & \dots & 0 \\ 0 & Q_2 & A_2 & 0 & \dots & 0 \\ \vdots & & \ddots & \ddots & & \\ 0 & \dots & & 0 & Q_m & A_m \end{bmatrix}$$

where $A_i = A$, $Q_i = DA$ for all i and all the remaining coefficients are zero (see [20] or section 1.4 below for the definition of the operations on $A^{(m)}$). Then $A^{(m)}$ is a quotient of the repetitive algebra $\hat{A}$ of A (see [20]). Since A is hereditary, the structure of the module category $\operatorname{mod} A^{(m)}$ is known (see section 3.1 below). As a first useful consequence, we show that the projective dimension of any indecomposable $A^{(m)}$-module is completely determined by its position inside the module category (see Proposition 17 below).

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In order to relate the tilting $A^{(m)}$-modules to the tilting objects in $\mathcal{C}_m(A)$, we need to check whether exceptional modules with a maximal number of summands are indeed tilting modules. We recall that, if C is a finite dimensional algebra, a C-module T is called *exceptional* if

(1) the projective dimension $\operatorname{pd} T = d$ of T is finite, and
(2) $\operatorname{Ext}^i_C(T, T) = 0$ for all $i \geq 1$.
An exceptional module T is called a *(generalized) tilting module* (see [17]) if moreover :
(3) there exists an exact sequence

$$0 \to C_C \to T_0 \to T_1 \to \cdots \to T_d \to 0$$

where each T_i is a direct sum of direct summands of T.

It is an important open problem whether, for an exceptional module T, having the number of isomorphism classes of indecomposable summands equal to the rank of the Grothendieck group of C, is sufficient for T to be tilting. This was first proven by Bongartz in case $\operatorname{pd} T = 1$, and the way he did it was to prove that, if T is exceptional, then there exists a module X such that $T \oplus X$ is a tilting module [9]. This latter statement (and hence the conjecture) were shown later for the case where C is representation-finite [23] (see also [13]). We prove here the analogue of Bongartz' result in another special case and deduce our first theorem.

Theorem 1 *Let A be a hereditary algebra over an algebraically closed field and $A^{(m)}$ be its m-th replicated algebra. Let T be a faithful exceptional $A^{(m)}$-module with $\operatorname{pd} T \leq m$, and such that the number of isomorphism classes of indecomposable summands of T equals the rank of the Grothendieck group. Then T is a tilting module.*

We then proceed to describe the m-cluster category $\mathcal{C}_m(A)$. By Happel's theorem [17], the derived category $\mathcal{D}^b(\operatorname{mod} A)$ is equivalent to the stable module category over the repetitive algebra $\hat{A}$ of A. The natural embedding of $\operatorname{mod} A^{(m)}$ into $\operatorname{mod} \hat{A}$ induces a functor π from $\operatorname{mod} A^{(m)}$ to the m-cluster category $\mathcal{C}_m(A)$. Defining the m-left part $\mathcal{L}_m(A^{(m)})$ of $A^{(m)}$ to consist of the indecomposable $A^{(m)}$-modules all of whose predecessors have projective dimension at most m and the functor π to be the composition

$$\pi : \operatorname{mod} A^{(m)} \hookrightarrow \operatorname{mod} \hat{A} \twoheadrightarrow \underline{\operatorname{mod}}\, \hat{A} \cong \mathcal{D}^b(\operatorname{mod} A) \twoheadrightarrow \mathcal{C}_m(A),$$

this leads to the second theorem.

Theorem 2 *Let A be a hereditary algebra over an algebraically closed field and $A^{(m)}$ be its m-th replicated algebra. The functor π induces a one-to-one correspondence between the non projective-injective modules lying in the m-left part $\mathcal{L}_m(A^{(m)})$ and the indecomposable objects in $\mathcal{C}_m(A)$.*

This is expressed by saying that $\mathcal{L}_m(A^{(m)})$ is an *exact fundamental domain* for the functor π (compare with [3]). We next characterize the tilting objects in $\mathcal{C}_m(A)$ in terms of the tilting $A^{(m)}$-modules. An exceptional $A^{(m)}$-module T is called an $\mathcal{L}_m -$ *exceptional* module if, whenever it is written in the form $T = T' \oplus P$, with P projective-injective, and T' having no projective-injective direct summand, then all the indecomposable summands of T' lie in $\mathcal{L}_m(A^{(m)})$.

Theorem 3 *Let A be a hereditary algebra over an algebraically closed field and $A^{(m)}$ be its m-th replicated algebra. There is a one-to-one correspondence between the $\mathcal{L}_m$-exceptional $A^{(m)}$-modules and the exceptional objects in $\mathcal{C}_m(A)$ given by $T = T' \oplus P \mapsto \pi(T')$.*

As a direct consequence of our Theorems 1 and 3, the above correspondence induces a one-to-one correspondence between the $\mathcal{L}_m$-tilting $A^{(m)}$-modules and the tilting objects in $\mathcal{C}_m(A)$.

Clearly, our theorems 2 and 3 generalize the main results of [3]. The proofs here are however different, and rest on the analysis of the projective dimension of the modules under consideration.

We now describe the contents of the paper. After a brief preliminary section, in which we fix the notations and recall the concepts needed in the paper, our section 2 is devoted to the analysis of the projective dimensions of the indecomposable modules over an infinite dimensional (but locally finite dimensional) quotient of the repetitive algebra, called the right repetitive algebra. These results are then applied to the m-replicated algebra in section 3, which culminates with the proof of Theorem 2. Section 4 is devoted to the proof of Theorem 1, and section 5 to the proof of Theorem 3.

1 Preliminaries

1.1 Notation

Throughout this paper, algebras are basic and connected over a fixed algebraically closed field. Given a locally finite dimensional algebra C (see [10]), we denote by $\operatorname{mod} C$ the category of finitely generated right C-modules and by $\operatorname{ind} C$ a full subcategory whose objects are a full set of representatives of the isomorphism classes of indecomposable C-modules. Whenever we say that a given C-module is indecomposable, we always mean implicitly that it belongs to $\operatorname{ind} C$. Throughout this paper, all subcategories of $\operatorname{mod} C$ are full, and so are identified with their object classes. Given a subcategory $\mathcal{C}$ of $\operatorname{mod} C$, we sometimes write $M \in \mathcal{C}$ to express that M is an object in $\mathcal{C}$. We denote by

$\operatorname{add}\mathcal{C}$ the subcategory of $\operatorname{mod} C$ having as objects the finite direct sums of objects in $\mathcal{C}$ and, if M is a module, we abbreviate $\operatorname{add}\{M\}$ as $\operatorname{add} M$. We denote the projective dimension of a module M as $\operatorname{pd} M$. The global dimension of C is denoted by $\operatorname{gl.dim} C$ and the quiver of C by $\mathcal{Q}_C$.

Given an algebra C, we denote by $\nu_C = - \otimes_C DC$ its Nakayama functor, and by τ_C its Auslander-Reiten translation. If M is a C-module, then its first syzygy $\Omega_C M$ is the kernel of a projective cover $P \to M$ in $\operatorname{mod} C$ and its first cosyzygy $\Omega_C^{-1} M$ is the cokernel of an injective envelope $M \to I$. For further facts and definitions needed on $\operatorname{mod} C$ and the Auslander-Reiten quiver $\Gamma(\operatorname{mod} C)$ of C, we refer the reader to [7,24]. For (minimal) approximations we refer to [6].

1.2 The m-left part

Let C be a (locally) finite dimensional algebra, and M, N be two indecomposable C-modules. A *path* from M to N in $\operatorname{ind} C$ is a sequence of non-zero morphisms

$$M = M_0 \xrightarrow{f_1} M_1 \xrightarrow{f_2} \cdots \xrightarrow{f_t} M_t = N$$

with all M_i in $\operatorname{ind} C$. Following [24], we denote the existence of such a path by $M \leq N$. We say that M is a *predecessor* of N (or that N is a *successor* of M).

More generally, if S_1 and S_2 are two sets of modules, we write $S_1 \leq S_2$ if every module in S_2 has a predecessor in S_1, every module in S_1 has a successor in S_2 and no module in S_2 has a successor in S_1 and no module in S_1 has a predecessor in S_2. The notation $S_1 < S_2$ stands for $S_1 \leq S_2$ and $S_1 \cap S_2 = \emptyset$.

Let $m \geq 1$. The *m-left part* $\mathcal{L}_m(C)$ of $\operatorname{mod} C$ is the full subcategory of $\operatorname{ind} C$ consisting of all indecomposable C-modules M such that if $L \leq M$, then $\operatorname{pd} L \leq m$.

Clearly, $\mathcal{L}_1(C)$ is the left part in the sense of [19].

1.3 The cluster category and the m-cluster category

Let A be a hereditary finite dimensional algebra, and F denote the endofunctor of $\mathcal{D}^b(\operatorname{mod} A)$ defined as the composition $\tau^{-1}[1]$, where τ is the Auslander-Reiten translation in $\mathcal{D}^b(\operatorname{mod} A)$ and $[1]$ is the shift functor. The *cluster category* $\mathcal{C}(A)$ (see [11]) has as objects the F-orbits of objects in $\mathcal{D}^b(\operatorname{mod} A)$ and

the morphisms are given by

$$\mathrm{Hom}_{\mathcal{C}(A)}(\tilde{X}, \tilde{Y}) = \oplus_{i\in\mathbb{Z}} \mathrm{Hom}_{\mathcal{D}^b(\mathrm{mod}\, A)}(X, F^i Y)$$

where X and Y are objects in $\mathcal{D}^b(\mathrm{mod}\, A)$ and $\tilde{X}$, $\tilde{Y}$ are their respective F-orbits. It is shown in [21] that $\mathcal{C}(A)$ is a triangulated category.

More generally, let $m \geq 1$ and F_m denote the endofunctor of $\mathcal{D}^b(\mathrm{mod}\, A)$ defined as the composition $\tau^{-1}[m]$. The *m-cluster category* $\mathcal{C}_m(A)$ (see [25]) has as objects the F_m-orbits of objects in $\mathcal{D}^b(\mathrm{mod}\, A)$ and the morphisms are given by

$$\mathrm{Hom}_{\mathcal{C}_m(A)}(\tilde{X}, \tilde{Y}) = \oplus_{i\in\mathbb{Z}} \mathrm{Hom}_{\mathcal{D}^b(\mathrm{mod}\, A)}(X, F_m^i Y)$$

where X and Y are objects in $\mathcal{D}^b(\mathrm{mod}\, A)$ and $\tilde{X}$, $\tilde{Y}$ are their respective F_m-orbits. Again, by [21], $\mathcal{C}_m(A)$ is a triangulated category. We refer to [25,22,28,8] for facts about the m-cluster category.

1.4 The repetitive algebra

Let C be a finite dimensional algebra. Its repetitive algebra $\hat{C}$ is the infinite matrix algebra

$$\hat{C} \quad = \quad \begin{bmatrix} \ddots & & & & 0 & \\ & C_{i-1} & & & & \\ & Q_i & C_i & & & \\ & & Q_{i+1} & C_{i+1} & & \\ & 0 & & & \ddots & \end{bmatrix}$$

where matrices have only finitely many non-zero coefficients, $C_i = C$ and $Q_i = {}_C DC_C$ for all $i \in \mathbb{Z}$, all the remaining coefficients are zero and multiplication is induced from the canonical isomorphisms $C \otimes_C DC \cong {}_C DC_C \cong DC \otimes_C C$ and the zero morphism $DC \otimes_C DC \to 0$, see [20]. Then $\hat{C}$ is an infinite dimensional, locally finite dimensional, self-injective algebra without identity. The importance of $\hat{C}$ in our case comes from the following result of [17].

Theorem 4 (Happel) *Let C be of finite global dimension, then the derived category $\mathcal{D}^b(\mathrm{mod}\, C)$ is equivalent, as a triangulated category, to the stable module category $\underline{\mathrm{mod}}\, \hat{C}$.*

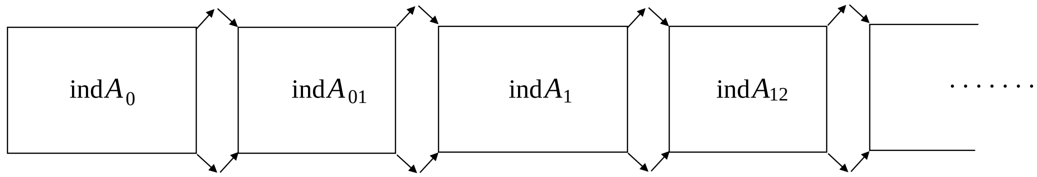


Fig. 1. Auslander-Reiten quiver of $\grave{A}$

2 The right repetitive algebra

2.1 Definition and description of the Auslander-Reiten quiver

Let C be a finite dimensional algebra. The *right repetitive algebra* $\grave{C}$ of C, introduced in [2], is the quotient of the repetitive algebra $\hat{C}$ of C defined by:

$$\grave{C} \quad = \quad \begin{bmatrix} C_0 & & & 0 \\ Q_1 & C_1 & & \\ & Q_2 & C_2 & \\ 0 & & \ddots & \ddots \end{bmatrix}$$

where, as in section 1.4, $C_i = C$ and $Q_i = {}_C DC_C$ for all $i \in \mathbb{Z}$.

Assume, from now on, that A is a hereditary algebra. The description of $\mathrm{mod}\,\grave{A}$ follows easily from [26,27,20,1] and can be summarized as follows.

Lemma 5 *(1) The standard embeddings* $\mathrm{ind}\,A_i \hookrightarrow \mathrm{ind}\,\grave{A}$ *(for* $i \geq 0$*) and* $\mathrm{ind}\,\grave{A} \hookrightarrow \mathrm{ind}\,\hat{A}$ *are full, exact, preserve indecomposable modules, almost split sequences and irreducible morphisms.*

(2) Under these embeddings, each $\mathrm{ind}\,A_i$ *is a full convex subcategory of* $\mathrm{ind}\,\grave{A}$*, and* $\mathrm{ind}\,\grave{A}$ *is a full convex subcategory of* $\mathrm{ind}\,\hat{A}$*. Moreover,* $\mathrm{ind}\,A_0$ *is closed under predecessors in* $\mathrm{ind}\,\grave{A}$*, and* $\mathrm{ind}\,\grave{A}$ *is closed under successors in* $\mathrm{ind}\,\hat{A}$*.*

In the sequel, we identify A with A_0 and each $\mathrm{ind}\,A_i$ with the corresponding full subcategory of $\mathrm{ind}\,\grave{A}$. Thus the Auslander-Reiten quiver $\Gamma(\mathrm{mod}\,\grave{A})$ of $\grave{A}$ can be described as follows (see Figure 1).

It starts with the Auslander-Reiten quiver $\Gamma(\mathrm{mod}\,A)$ of $A_0 = A$. Then projective-injective modules start to appear; such a projective-injective module has its socle corresponding to a point in the quiver $\mathcal{Q}_A$ of A, and its top corresponding to a point in the quiver of A_1. Next occurs a part denoted by $\mathrm{ind}\,A_{01}$ where indecomposables contain at the same time simple composition factors from $A_0 = A$, and simple composition factors from A_1. When all

projective-injectives whose socles correspond to points in the quiver of A_0 have appeared, we reach the projective A_1-modules and thus the Auslander-Reiten quiver $\Gamma(\mathrm{mod}\, A_1)$ of A_1. The situation then repeats itself.

This repetition is effected by means of the Nakayama functor, whose action is described as follows. For a point a (or an arrow α) in the ordinary quiver of A, denote by a_i (or α_i) the corresponding point (or arrow, respectively) in the quiver of A_i. Let thus M be an $\hat{A}$-module, considered as a representation. Then $\nu_{\hat{A}}\, M(a_i) = M(a_{i-1})$ and $\nu_{\hat{A}}\, M(\alpha_i) = M(\alpha_{i-1})$.

Observe that all injective $\hat{A}$-modules are projective, and that those projective $\hat{A}$-modules which are not injective are just the projective A-modules.

2.2 *Example*

Since the former picture is the basis for our intuition, we give here an example.

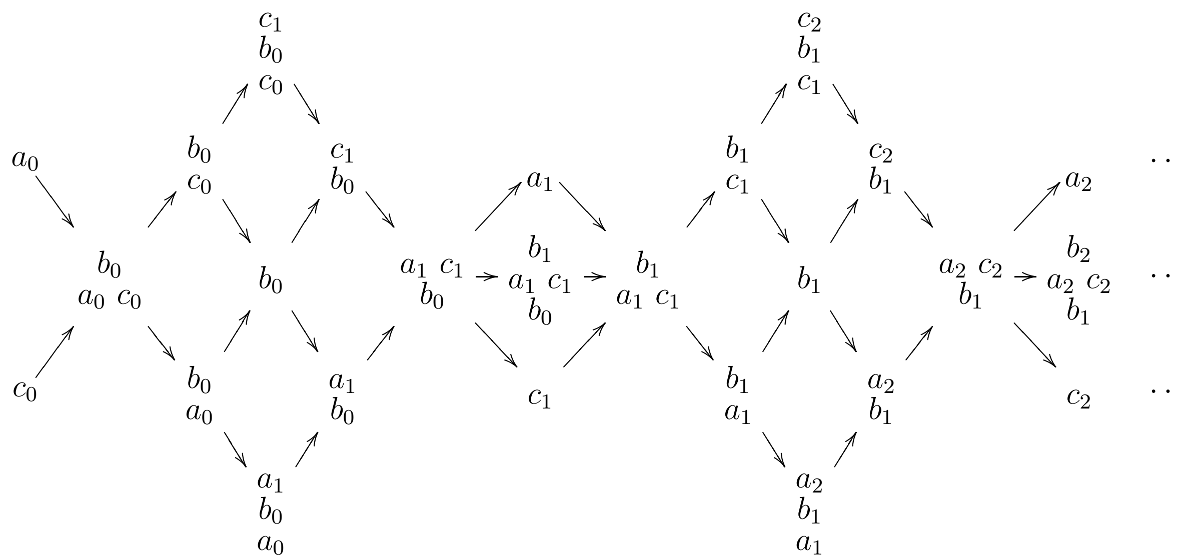
Let A be given by the quiver

$$\begin{tikzcd}
a & \\
& b \arrow[ul, "\alpha"'] \arrow[dl, "\beta"] \\
c &
\end{tikzcd}$$

then the quiver of $\hat{A}$ is

$$\begin{tikzcd}
a_0 & & a_1 \arrow[dl, "\lambda_1"'] & & a_2 \arrow[dl, "\lambda_2"'] & & \cdots & \\
& b_0 \arrow[ul, "\alpha_0"'] \arrow[dl, "\beta_0"] & & b_1 \arrow[ul, "\alpha_1"'] \arrow[dl, "\beta_1"] & & b_2 \arrow[ul, "\alpha_2"'] \arrow[dl, "\beta_2"] & & \cdots \\
c_0 & & c_1 \arrow[ul, "\mu_1"] & & c_2 \arrow[ul, "\mu_2"] & & \cdots &
\end{tikzcd}$$

bound by the relations $\lambda_{i+1}\beta_i = 0$, $\mu_{i+1}\alpha_i = 0$, $\alpha_{i+1}\lambda_{i+1} = \beta_{i+1}\mu_{i+1}$, for all $i \geq 0$. Then $\Gamma(\mathrm{mod}\, \hat{A})$ is as follows:



where modules are represented by their Loewy series.

2.3 Injective envelopes and projective covers

The following lemma is inspired from a well-known result about symmetric algebras.

Lemma 6 *Let L be an indecomposable $\grave{A}$-module and*

$$0 \to L \xrightarrow{f} I \xrightarrow{g} N \to 0$$

a short exact sequence of $\grave{A}$-modules with $L \xrightarrow{f} I$ an injective envelope, I projective-injective and $N \neq 0$. Then

(1) $I \xrightarrow{g} N$ is a projective cover in $\operatorname{mod} \grave{A}$ *and*
(2) N is indecomposable.

PROOF. (1) Let $p : P(N) \to N$ be a projective cover in $\operatorname{mod} \grave{A}$ and let $K = \operatorname{Ker} p$. Then there exists a projective $\grave{A}$-module P' and a commutative

diagram with exact rows and columns, where i is a section.

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & K \arrow[r] \arrow[d] & P(N) \arrow[r, "p"] \arrow[d, "i"'] & N \arrow[r] \arrow[d, "\mathrm{id}"'] & 0 \\
0 \arrow[r] & L \arrow[r, "f"] \arrow[d] & I \arrow[r, "g"] \arrow[d] & N \arrow[r] & 0 \\
 & P' \arrow[r, "\mathrm{id}"] \arrow[d] & P' \arrow[d] & & \\
 & 0 & 0 & &
\end{tikzcd}$$

Hence P' is also injective. Then $L \cong K \oplus P'$. Since L is indecomposable and not projective-injective it follows that $L \cong K$. Thus $P(N) = I$.

(2) Suppose that $N = N_1 \oplus N_2$. Then $I \cong P(N) \cong P(N_1) \oplus P(N_2)$ and there exists an induced direct sum decomposition of the kernel $L = L_1 \oplus L_2$. Since L is indecomposable, we may assume that $L = L_1$ and $L_2 = 0$. But then $P(N_2) \cong N_2$ so that we have a commutative diagram with exact rows.

$$\begin{tikzcd}
0 \arrow[r] & L \arrow[r, "f"] \arrow[d, "\mathrm{id}"'] & I \arrow[r, "g"] \arrow[d, "\cong"'] & N \arrow[r] \arrow[d, "\mathrm{id}"'] & 0 \\
0 \arrow[r] & L \arrow[r] & P(N_1) \oplus N_2 \arrow[r, "{\left(\begin{smallmatrix} k & 0 \\ 0 & \mathrm{id} \end{smallmatrix}\right)}"] & N_1 \oplus N_2 \arrow[r] & 0
\end{tikzcd}$$

which contradicts the hypothesis that $L \xrightarrow{f} I$ is an injective envelope. $\square$

Corollary 7 *Let L be an indecomposable $\grave{A}$-module and*

$$0 \longrightarrow L \xrightarrow{f^0} I^0 \xrightarrow{f^1} I^1 \longrightarrow \cdots \xrightarrow{f^k} I^k$$

be a minimal injective coresolution in $\operatorname{mod} \grave{A}$, *with all I^j projective-injective and $N = \operatorname{Coker} f^k \neq 0$. Then:*

(1) $I^0 \xrightarrow{f^1} I^1 \longrightarrow \cdots \xrightarrow{f^k} I^k \longrightarrow N \longrightarrow 0$ *is a minimal projective resolution in* $\operatorname{mod} \grave{A}$,
(2) N is indecomposable and
(3) for all $j \leq k$, the j-th cosyzygy of L is isomorphic to the $(k+1-j)$-th syzygy of N, that is

$$\Omega_{\grave{A}}^{-j} L \;\cong\; \Omega_{\grave{A}}^{(k+1)-j} N.$$

PROOF. By induction on k using Lemma 6. $\square$

Lemma 8 *Let M be an indecomposable $\grave{A}$-module which does not lie in* $\operatorname{ind} A$. *Then the projective cover of M is projective-injective.*

PROOF. We may clearly assume that M is not projective-injective. Since $M \notin \operatorname{ind} A$, we have

$$\overline{\operatorname{Hom}}_{\grave{A}}(A_A, M) \cong \overline{\operatorname{Hom}}_{\hat{A}}(A_A, M) = 0,$$

where the last equality follows from Happel's Theorem (see Theorem 4 above) and from the structure of morphisms in the derived category. Therefore, any non-zero morphism in $\operatorname{Hom}_{\grave{A}}(A_A, M)$ must factor through an injective $\grave{A}$-module which is also projective. The statement follows. $\square$

Lemma 9 *Let M be an indecomposable $\grave{A}$-module, then*

$$\Omega_{\grave{A}}^{-1} \, \tau_{\grave{A}}^{-1} \, M \;\cong\; \tau_{\grave{A}}^{-1} \, \Omega_{\grave{A}}^{-1} \, M$$

PROOF. For any indecomposable $\grave{A}$-module M, we have $\Omega_{\grave{A}}^{-1} M = \Omega_{\hat{A}}^{-1} M$, because injective $\grave{A}$-modules are also $\hat{A}$-injective, and $\tau_{\grave{A}}^{-1} M = \tau_{\hat{A}}^{-1} M$. The statement follows from the fact that $\tau_{\hat{A}}^{-1} = \Omega_{\hat{A}}^{-2} \nu_{\hat{A}}^{-1}$ (see [7, IV.3.7 p.126]). $\square$

2.4 Projective dimension

We are now able to prove the main result of this section.

Theorem 10 *Let M be an indecomposable $\grave{A}$-module and let $k \geq 1$. Then* $\operatorname{pd} M = k$ *if and only if there exists an indecomposable A-module N such that $M \cong \tau_{\grave{A}}^{-1} \, \Omega_{\grave{A}}^{-(k-1)} \, N$.*

PROOF. We prove the statement by induction on k. Suppose $k = 1$. If $M \cong \tau_{\grave{A}}^{-1} N$ for some N in $\operatorname{ind} A$ then it follows from [3, Cor 5 and 6] that M lies in the left part $\mathcal{L}_{\grave{A}}$ of $\grave{A}$, and thus, $\operatorname{pd} M \leq 1$. On the other hand, $M \cong \tau_{\grave{A}}^{-1} N$ implies that M is not projective, and hence $\operatorname{pd} M = 1$.

Conversely, assume that M is not of the form $\tau_{\grave{A}}^{-1} N$, for some $N \in \operatorname{ind} A$. If M is projective then $\operatorname{pd} M = 0 \neq 1$. Otherwise, by Lemma 8, in the short exact sequence

$$0 \to \Omega_{\grave{A}} \, M \to P \xrightarrow{f} M \to 0, \tag{1}$$

where f is a projective cover, the module P is projective-injective and, by [3, Prop. 7], $\Omega_{\grave{A}}\, M$ is not a projective A-module. Clearly, $\Omega_{\grave{A}}\, M$ cannot be projective-injective, since (1) is not split. Hence $\operatorname{pd} M \neq 1$.

Suppose now $k > 1$. Let $M \in \operatorname{ind} \grave{A}$ such that $\operatorname{pd} M = k$. Since $k > 1$, then $M \notin \operatorname{ind} A$ and, by Lemma 8, the projective cover of M in $\operatorname{mod} \grave{A}$ is projective-injective. Let $X = \Omega_{\grave{A}}\, M$. Then $\operatorname{pd} X = k - 1$. By the dual of Lemma 6, X is indecomposable. Now, by the induction hypothesis, $X = \tau_{\grave{A}}^{-1} Y$ for some $Y \in \Omega_{\grave{A}}^{-(k-2)} \operatorname{ind} A$. Moreover, by Lemma 6, $M = \Omega_{\grave{A}}^{-1} X$ and therefore

$$M = \Omega_{\grave{A}}^{-1}\, X = \Omega_{\grave{A}}^{-1}\, \tau_{\grave{A}}^{-1}\, Y \cong \tau_{\grave{A}}^{-1}\, \Omega_{\grave{A}}^{-1}\, Y$$

lies in $\tau_{\grave{A}}^{-1}\, \Omega_{\grave{A}}^{-(k-1)} \operatorname{ind} A$, as required.

Conversely, suppose that $M = \tau_{\grave{A}}^{-1}\, \Omega_{\grave{A}}^{-(k-1)}\, N$ for some $N \in \operatorname{ind} A$. By Corollary 7, $X = \tau_{\grave{A}}^{-1}\, \Omega_{\grave{A}}^{-(k-2)}\, N$ is indecomposable. Furthermore, $M = \Omega_{\grave{A}}^{-1}\, X$ and there is a short exact sequence

$$0 \to X \xrightarrow{f} I \xrightarrow{g} M \to 0,$$

with f an injective envelope in $\operatorname{mod} \grave{A}$. Thus I is projective-injective and, by Lemma 6, g is a projective cover; hence $\operatorname{pd} M = \operatorname{pd} X + 1$. But by induction, the projective dimension of X is $k - 1$, and thus $\operatorname{pd} M = k$. This completes the proof. $\square$

Recall that a component of the Auslander-Reiten quiver of $\grave{A}$ is called *transjective* if it does not contain cycles in $\operatorname{ind} \grave{A}$. A full connected subquiver Σ in an Auslander-Reiten component is called a subsection if every path in Σ is sectional.

Definition 11 *A* right stable slice Σ *in* $\operatorname{ind} \grave{A}$ *is a connected convex subsection in a transjective component of the Auslander-Reiten quiver of* $\operatorname{mod} \grave{A}$ *which intersects each right stable* $\tau_{\grave{A}}$*-orbit in that component.*

For each k greater than or equal to zero, we set $\Sigma_k = \left\{ \Omega_{\grave{A}}^{-k}\, P_x \mid x \in (\mathcal{Q}_A)_0 \right\}$. Notice that Σ_0 is just the set of projective A-modules and that $\Sigma_k = \Omega^{-k}(\Sigma_0)$.

Lemma 12 *For each* $k \geq 0$*,* Σ_k *is a right stable slice in* $\operatorname{ind} \grave{A}$*.*

PROOF. By induction on k. Clearly Σ_0,which is the set of all indecomposable projective A-modules, is a right stable slice. Also, by [3, Cor. 6], $\Sigma_1 = \{\Omega_{\grave{A}}^{-1}\, P_x \mid x \in (\mathcal{Q}_A)_0\}$ is a right stable slice. Assume now that $k > 1$. If k is even we have $\Omega_{\grave{A}}^{-k} = \Omega_{\grave{A}}^{-2(k/2)} = \tau_{\grave{A}}^{-k/2}\, \nu_{\grave{A}}^{k/2}$, by [7, IV.3.7 p.126]. Since both $\tau_{\grave{A}}$ and

$\nu_{\grave{A}}$ preserve right stable slices, the statement follows from $\Sigma_k = \tau_{\grave{A}}^{-k/2}\,\nu_{\grave{A}}^{k/2}\,\Sigma_0$. If, on the other hand, k is odd, then the statement follows similarly from $\Sigma_k = \tau_{\grave{A}}^{-(k-1)/2}\,\nu_{\grave{A}}^{(k-1)/2}\,\Sigma_1$. $\square$

We now show that the right stable slices Σ_k partition $\underline{\mathrm{ind}}\,\grave{A}$ into regions of constant projective dimension. It is useful to note that, if $L \leq M$ in $\mathrm{ind}\,\grave{A}$, then $\Omega_{\grave{A}}\,L \leq \Omega_{\grave{A}}\,M$ and $\Omega_{\grave{A}}^{-1}\,L \leq \Omega_{\grave{A}}^{-1}\,M$.

Corollary 13 *Let M be an indecomposable $\grave{A}$-module which is not projective-injective and let $k \geq 1$. Then*

$$\mathrm{pd}\,M = k \quad \textit{if and only if} \quad \Sigma_{k-1} < M \leq \Sigma_k$$

PROOF. Suppose $\mathrm{pd}\,M = k$. By Theorem 10, there exists $N \in \mathrm{ind}\,A$ such that

$$M = \tau_{\grave{A}}^{-1}\,\Omega_{\grave{A}}^{-(k-1)}\,N = \Omega_{\grave{A}}^{-(k-1)}\left(\tau_{\grave{A}}^{-1}\,N\right).$$

Now, by [3], we have $\Sigma_0 < \tau_{\grave{A}}^{-1}\,N \leq \Sigma_1$. Since $\Sigma_{k-1} = \Omega_{\grave{A}}^{-(k-1)}\,\Sigma_0$ and $\Sigma_k = \Omega_{\grave{A}}^{-(k-1)}\,\Sigma_1$, by definition, this implies $\Sigma_{k-1} < M \leq \Sigma_k$.

Conversely, assume that $\Sigma_{k-1} < M \leq \Sigma_k$. Since, by Corollary 7, we have $\Sigma_0 = \Omega_{\grave{A}}^{k-1}\Sigma_{k-1}$ and $\Sigma_1 = \Omega_{\grave{A}}^{k-1}\Sigma_k$, applying $\Omega_{\grave{A}}^{k-1}$ to the former inequality yields $\Sigma_0 < \Omega_{\grave{A}}^{k-1}\,M \leq \Sigma_1$. Then, by [3, Cor. 5], it follows that $\mathrm{pd}\,\Omega_{\grave{A}}^{k-1}\,M = 1$, whence $\mathrm{pd}\,M = k$. $\square$

The right repetitive algebra $\grave{A}$ thus provides an example of an (infinite-dimensional) algebra such that every indecomposable $\grave{A}$-module has finite projective dimension, but such that the finitistic projective dimension of $\grave{A}$ is infinite. (We recall that the finitistic projective dimension of an algebra is the supremum of the projective dimensions of those modules which have finite projective dimension.)

3 The m-replicated algebra $A^{(m)}$

3.1 Definition and description of the Auslander-Reiten quiver

Let $m \geq 1$ and C be a finite dimensional algebra. The *m-replicated algebra* $C^{(m)}$ of C is the quotient of the right repetitive algebra $\grave{C}$ (hence of the

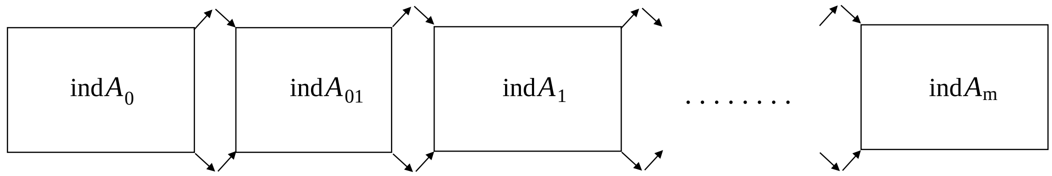


Fig. 2. Auslander-Reiten quiver of $A^{(m)}$

repetitive algebra $\hat{C}$) defined by

$$C^{(m)} = \begin{bmatrix} C_0 & 0 & \dots\dots & \dots & 0 \\ Q_1 & C_1 & 0 & \dots\ \dots & 0 \\ 0 & Q_2 & C_2 & 0\ \dots & 0 \\ \vdots & & \ddots & \ddots & \\ 0 & \dots & 0 & Q_m & C_m \end{bmatrix}.$$

If $m = 1$, then $C^{(1)}$ is called the *duplicated algebra* of C (see [3]). It is shown in [4] that

$$m + \mathrm{gl.dim}\, C \leq \mathrm{gl.dim}\, C^{(m)} \leq (m+1)\mathrm{gl.dim}\, C + m.$$

Assume from now on that A is hereditary. The description of $A^{(m)}$ follows from [26,27,1] and can be summarized as follows.

Lemma 14 *(1) The standard embeddings* $\mathrm{ind}\, A_i \hookrightarrow \mathrm{ind}\, A^{(m)}$ *(where* $0 \leq i \leq m$*) and* $\mathrm{ind}\, A^{(m)} \hookrightarrow \mathrm{ind}\, \grave{A}$ *are full, exact, preserve indecomposable modules, almost split sequences and irreducible morphisms.*

(2) Under the above embeddings, each $\mathrm{ind}\, A_i$ *is a full convex subcategory of* $\mathrm{ind}\, A^{(m)}$*, and* $\mathrm{ind}\, A^{(m)}$ *is a full convex subcategory of* $\mathrm{ind}\, \grave{A}$*. Moreover,* $\mathrm{ind}\, A_0$ *is closed under predecessors and* $\mathrm{ind}\, A_m$ *is closed under successors in* $\mathrm{ind}\, A^{(m)}$*.*

In the sequel, we identify A with A_0 and each $\mathrm{ind}\, A_i$ with the corresponding full subcategory of $\mathrm{ind}\, A^{(m)}$, and $\mathrm{ind}\, A^{(m)}$ with the corresponding full subcategory of $\mathrm{ind}\, \grave{A}$. The Auslander-Reiten quiver of $A^{(m)}$ can be deduced from that of $\grave{A}$ (see Figure 2).

3.2 Projective dimension

Since we are interested in the projective dimension of indecomposable $A^{(m)}$-modules, we need to study projective covers. In our situation, Lemma 8 becomes the following lemma.

Lemma 15 *(1) Let M be an indecomposable $A^{(m)}$-module which does not lie in* $\operatorname{ind} A$*. Then its projective cover in* $\operatorname{mod} A^{(m)}$ *is projective-injective and coincides with its projective cover in* $\operatorname{mod} \grave{A}$*.*

(2) Let M be an indecomposable $A^{(m)}$-module which does not lie in $\operatorname{ind} A_m$*. Then its injective envelope in* $\operatorname{mod} A^{(m)}$ *is projective-injective and coincides with its injective envelope in* $\operatorname{mod} \grave{A}$*.*

PROOF. Since (2) is dual to (1), we only prove (1). The case where M is projective-injective is trivial. Assume that M is injective but not projective. Then $M \in \operatorname{ind} A_m$ and therefore $\operatorname{Hom}_{A^{(m)}}(A, M) = 0$ (because the supports of A and M are disjoint) and the statement follows. In the case where M is not injective one can use the same argument as in the proof of Lemma 8 to show that the projective cover of M is projective-injective. It coincides with its projective cover in $\operatorname{mod} \grave{A}$ because $\operatorname{ind} A^{(m)}$ is closed under predecessors in $\operatorname{ind} \grave{A}$. $\square$

We now show that, for each $k \leq m$ the right stable slice Σ_k of Lemma 12 consists of $A^{(m)}$-modules.

Lemma 16 *We have $\Sigma_m \subset \operatorname{ind} A^{(m)}$.*

PROOF. We prove by induction on $k \leq m$ that $\Sigma_k \leq \nu_{\grave{A}}^k \Sigma_0$. If $k = 1$, then by [3], $\Sigma_1 = \Omega_{\grave{A}}^{-1} \Sigma_0 \leq \nu_{\grave{A}} \Sigma_0$. Assume now that, for some $k < m$, we have $\Sigma_k \leq \nu_{\grave{A}}^k \Sigma_0$. Since $\nu_{\grave{A}} = \tau_{\grave{A}} \Omega_{\grave{A}}^{-2}$, we have

$$\nu_{\grave{A}}^{k+1} \Sigma_0 \;=\; \tau_{\grave{A}} \Omega_{\grave{A}}^{-2} \nu_{\grave{A}}^k \Sigma_0 \;\geq\; \tau_{\grave{A}} \Omega_{\grave{A}}^{-2} \Sigma_k \;=\; \tau_{\grave{A}} \Omega_{\grave{A}}^{-1} \Sigma_{k+1}.$$

On the other hand, if $X \in \operatorname{ind} \grave{A}$ is not a projective-injective module then $\operatorname{Hom}_{\grave{A}}(\tau_{\grave{A}}^{-1} X, \Omega_{\grave{A}}^{-1} X) \neq 0$ so that $\tau_{\grave{A}}^{-1} X \leq \Omega_{\grave{A}}^{-1} X$ and then $X \leq \tau_{\grave{A}} \Omega_{\grave{A}}^{-1} X$. Therefore $\nu_{\grave{A}}^{k+1} \Sigma_0 \geq \Sigma_{k+1}$, establishing our claim. The statement of the lemma then follows from $\Sigma_m \leq \nu_{\grave{A}}^m \Sigma_0 = A_m$. $\square$

We are now able to prove the statement corresponding to Theorem 10 and Corollary 13 in $\operatorname{mod} A^{(m)}$.

Proposition 17 *Let M be an indecomposable $A^{(m)}$-module which is not projective and let k be such that $1 \leq k \leq m$. The following are equivalent:*

(1) $\operatorname{pd} M = k$,
(2) there exists $N \in \operatorname{ind} A$ such that $M \cong \tau_{A^{(m)}}^{-1} \Omega_{A^{(m)}}^{-(k-1)} N$,
(3) $\Sigma_{k-1} < M \leq \Sigma_k$.

PROOF. Since $\operatorname{ind} A^{(m)}$ is closed under predecessors in $\operatorname{ind} \grave{A}$, then a minimal projective resolution of an indecomposable $A^{(m)}$-module X in $\operatorname{mod} A^{(m)}$ is also a minimal projective resolution of X in $\operatorname{mod} \grave{A}$ (and, in particular, the respective syzygies coincide). Therefore, by Theorem 10, (1) holds if and only if there exists $N \in \operatorname{ind} A$ such that

$$M \cong \tau_{\grave{A}}^{-1}\, \Omega_{\grave{A}}^{-(k-1)}\, N \cong \tau_{A^{(m)}}^{-1}\, \Omega_{A^{(m)}}^{-(k-1)}\, N.$$

Finally, the equivalence of (1) and (3) follows from Lemma 16 and Corollary 13. $\square$

Corollary 18 *Let M be an indecomposable $A^{(m)}$-module which is not projective-injective. The following are equivalent:*

(1) $M \in \mathcal{L}_m(A^{(m)})$,
(2) $M \in \cup_{k=1}^{m} \Omega_{A^{(m)}}^{-(k-1)}(\operatorname{ind} A) \cup \Sigma_m$,
(3) $M \leq \Sigma_m$.

PROOF. The equivalence of (1) and (3) follows from Proposition 17. We show that (1) implies (2). Let $M \in \mathcal{L}_m(A^{(m)})$, then $\operatorname{pd} M = k \leq m$. By Proposition 17, there exists $N \in \operatorname{ind} A$ such that

$$M \;=\; \tau_{\grave{A}}^{-1}\, \Omega_{\grave{A}}^{-(k-1)}\, N \;\cong\; \Omega_{\grave{A}}^{-(k-1)}\, \tau_{\grave{A}}^{-1}\, N \;\cong\; \Omega_{A^{(m)}}^{-(k-1)}\, \tau_{A^{(m)}}^{-1}\, N.$$

Thus, if N is not an injective A-module, then $M \in \Omega_{A^{(m)}}^{-(k-1)}(\operatorname{ind} A)$. On the other hand, if N is an injective A-module, then $\tau_{A^{(m)}}^{-1}\, N \in \Omega_{A^{(m)}}^{-1}\, \Sigma_0$ (by [3]) so that $M \in \Omega_{A^{(m)}}^{-(k-1)}\, \Omega_{A^{(m)}}^{-1}\, \Sigma_0 \;=\; \Omega_{A^{(m)}}^{-k}\, \Sigma_0 \;=\; \Sigma_k$. Thus $M \in \Sigma_m$ if $k = m$ and $M \in \cup_{k=1}^{m} \Omega_{A^{(m)}}^{-(k-1)}(\operatorname{ind} A)$ otherwise.

Finally we prove that (2) implies (3). We assume $M \in \cup_{k=1}^{m} \Omega_{A^{(m)}}^{-(k-1)}(\operatorname{ind} A) \cup \Sigma_m$ and claim that $M \leq \Sigma_m$. If $M \in \Sigma_m$, there is nothing to prove. Otherwise, there exist $N \in \operatorname{ind} A$ and $k \leq m$ such that $M = \Omega_{A^{(m)}}^{-(k-1)}\, N$. Now $N \leq \Sigma_1 = \Omega_{A^{(m)}}^{-1} \Sigma_0$ so $M \leq \Omega_{A^{(m)}}^{-(k-1)}\, \Omega_{A^{(m)}}^{-1}\, \Sigma_0 \;=\; \Omega_{A^{(m)}}^{-k}\, \Sigma_0 \;=\; \Sigma_k \leq \Sigma_m$. $\square$

3.3 Exact fundamental domain

One very easy consequence of Corollary 18, Theorem 10 and Corollary 13 is that $\mathcal{L}_m(A^{(m)}) = \mathcal{L}_m(\grave{A})$. Another one is the following corollary.

Corollary 19 *The embedding functor*

$$\operatorname{add} \mathcal{L}_m(A^{(m)}) \hookrightarrow \operatorname{mod} A^{(m)} \hookrightarrow \operatorname{mod} \grave{A} \hookrightarrow \operatorname{mod} \hat{A}$$

is full, exact and preserves indecomposable modules, irreducible morphisms and almost split sequences.

We are now able to prove Theorem 2. Since A is hereditary, we have, by [17], an equivalence $\mathcal{D}^b(\operatorname{mod} A) \cong \underline{\operatorname{mod}}\, \hat{A}$ of triangulated categories. Let

$$\pi : \operatorname{mod} A^{(m)} \hookrightarrow \operatorname{mod} \grave{A} \hookrightarrow \operatorname{mod} \hat{A} \twoheadrightarrow \underline{\operatorname{mod}}\, \hat{A} \cong \mathcal{D}^b(\operatorname{mod} A) \twoheadrightarrow \mathcal{C}_m(A).$$

be the canonical functor (where $\mathcal{C}_m(A)$ is the m-cluster category, see section 1.3). We define an *exact fundamental domain* for π to be a full convex subcategory of $\operatorname{ind} \hat{A}$ which contains exactly one point of each fibre $\pi^{-1}(X)$, with X an indecomposable object in $\mathcal{C}_m(A)$.

Theorem 20 *The functor π induces a one-to-one correspondence between the non projective-injective modules in $\mathcal{L}_m(A^{(m)})$ and the indecomposable objects in $\mathcal{C}_m(A)$. In particular, $\mathcal{L}_m(A^{(m)})$ is an exact fundamental domain for π.*

PROOF. Since $\mathcal{L}_m(A^{(m)})$ is a full convex subcategory of $\operatorname{ind} A^{(m)}$ and $\operatorname{ind} \grave{A}$, it is also convex inside $\operatorname{ind} \hat{A}$. Furthermore, the non projective-injective modules in $\mathcal{L}_m(A^{(m)})$ are just the modules in

$$\cup_{k=1}^{m} \Omega_{A^{(m)}}^{-(k-1)} (\operatorname{ind} A) \cup \Sigma_m \ = \ \cup_{k=1}^{m} \Omega_{\hat{A}}^{-(k-1)} (\operatorname{ind} A) \cup \Sigma_m .$$

The statement follows from the definition of $\mathcal{C}_m(A)$ and the fact that $\Omega_{\hat{A}}^{-1}$ corresponds to the shift [1] under the equivalence $\underline{\operatorname{mod}}\, \hat{A} \cong \mathcal{D}^b(\operatorname{mod} A)$. $\square$

4 Tilting modules over the replicated algebra

4.1 Definitions and preparatory results

Let C be a finite dimensional algebra, and T be a C-module. We say that T is an *exceptional module* if

(1) $\operatorname{pd} T = d < \infty$
(2) $\operatorname{Ext}_C^i(T, T) = 0$ for all $i \geq 1$
An exceptional module T is called a *(generalized) tilting module* if moreover
(3) there exists an exact sequence

$$0 \to C_C \to T_0 \to T_1 \to \cdots \to T_d \to 0$$

where each $T_i \in \operatorname{add} T$ for all i.

It is useful to observe that, if T is an exceptional module which is faithful, then any projective-injective indecomposable C-module P is a direct summand of T: indeed, since T is faithful, there exists a monomorphism $C_C \hookrightarrow T_0$ with $T_0 \in \operatorname{add} T$, which, when composed with the inclusion $P \hookrightarrow C_C$ yields an inclusion $P \hookrightarrow T_0$ which splits, because P is injective. In particular, if T is a tilting C-module, then any projective-injective indecomposable C-module is a direct summand of T.

An exceptional module T is said to be *basic* (or *multiplicity free*) if, whenever $T = \oplus_{i=1}^{n} T_i$ where all the T_i are indecomposable, we have $T_i \neq T_j$ for $i \neq j$. It is well-known that, if T is a basic tilting C-module, then the number of its indecomposable summands is equal to the rank of the Grothendieck group of C, see [17].

For the rest of this section, we let as before A be a hereditary algebra and $m \geq 1$, and $A^{(m)}$ denote the m-replicated algebra of A.

Lemma 21 *Let T be an exceptional $A^{(m)}$-module having all projective-injective indecomposable modules as direct summands and let M be an $A^{(m)}$-module. Assume that M has a projective-injective injective envelope. Then a minimal left* $\operatorname{add} T$*-approximation of M is a monomorphism.*

PROOF. Let $f_0 : M \to T_0$ be a minimal left $\operatorname{add} T$-approximation and $g : M \to I$ be an injective envelope. Since I is projective-injective, it lies in $\operatorname{add} T$, hence there exists $h : T_0 \to I$ such that $g = h\, f_0$. Since g is injective, so is f_0. □

Corollary 22 *An exceptional $A^{(m)}$-module T is faithful if and only if it has all projective-injective indecomposable $A^{(m)}$-modules as direct summands.*

PROOF. We have already shown the necessity. Conversely, assume any projective-injective indecomposable $A^{(m)}$-module to be a summand of T. By Lemma 21, a minimal left $\operatorname{add} T$-approximation of $A_{A^{(m)}}$ is a monomorphism. Therefore, there exists a monomorphism ${A^{(m)}}_{A^{(m)}} \hookrightarrow T_0$ with $T_0 \in \operatorname{add} T$. □

4.2 Tilting modules

We shall prove that, if T is a faithful exceptional $A^{(m)}$-module, all of whose non projective-injective summands lie in $\operatorname{add} \mathcal{L}_m(A^{(m)})$, then there exists an $A^{(m)}$-module X such that $T \oplus X$ is a tilting module.

Clearly, if $A^{(m)}$ (or, equivalently, A) is representation-finite, then this follows from [23]. We may then assume without loss of generality that A is representation-infinite.

Lemma 23 *Assume that A is representation-infinite, that T is a faithful exceptional $A^{(m)}$-module and that $M \in \mathcal{L}_m(A^{(m)})$. Then*

(1) the injective envelope of M is projective-injective, and
(2) a minimal left $\operatorname{add} T$-approximation of M is injective.

PROOF. By Lemma 21, it suffices to prove (1). Since, by Corollary 18, we have $M \in \mathcal{L}_m(A^{(m)})$ if and only if $M \leq \Sigma_m$, it suffices, by Lemma 14, to show that Σ_m contains no A_m-module. We first assume that $m = 2l$ is even, then

$$\Sigma_m \;=\; \Omega^{-m}_{A^{(m)}}\,\Sigma_0 \;=\; \Omega^{-2l}_{A^{(m)}}\,\Sigma_0 \;=\; \tau^{-l}_{A^{(m)}}\,(\nu^{l}_{A^{(m)}}\,\Sigma_0).$$

Since $\nu^{l}_{A^{(m)}}\,\Sigma_0$ is the set of projective A_l-modules, the set Σ_m consists of postprojective A_l-modules (here, we are using the assumption that $A^{(m)}$ is representation-infinite). Since $l \neq m$, this shows that $\Sigma_m \cap \operatorname{ind} A_m = \emptyset$. Assume now that $m = 2l+1$ is odd, then

$$\Sigma_m \;=\; \Omega^{-1}_{A^{(m)}}\,\Sigma_{2l} \;=\; \Omega^{-1}_{A^{(m)}}\,\tau^{-l}_{A^{(m)}}\,\nu^{l}_{A^{(m)}}\,\Sigma_0 \;=\; \tau^{-l}_{A^{(m)}}\,\Omega^{-1}_{A^{(m)}}\,(\nu^{l}_{A^{(m)}}\,\Sigma_0).$$

But $\nu^{l}_{A^{(m)}}\,\Sigma_0$ is the set of all indecomposable projective A_l-modules and, since A is representation-infinite, this implies $\Omega^{-1}_{A^{(m)}}\,(\nu^{l}_{A^{(m)}}\,\Sigma_0) \subset \operatorname{ind} A_{l,l+1}$ (in the notation of section 2.1 and section 3.1). We thus get again $\Sigma_m \cap \operatorname{ind} A_m = \emptyset$. $\square$

Remark 24 *The statement of Lemma 23 is false in the representation-finite case: let A be the path algebra of the quiver $a \leftarrow b$, then the Auslander-Reiten quiver of $A^{(1)}$ is:*

$$\begin{tikzcd}
a_0 \arrow[dr] & & b_0 \arrow[dr] & & a_1 \arrow[dr] & & b_1 \\
& \begin{smallmatrix} b_0 \\ a_0 \end{smallmatrix} \arrow[ur] \arrow[dr] & & \begin{smallmatrix} a_1 \\ b_0 \end{smallmatrix} \arrow[ur] \arrow[r] & \begin{smallmatrix} b_1 \\ a_1 \\ b_0 \end{smallmatrix} \arrow[r] & \begin{smallmatrix} b_1 \\ a_1 \end{smallmatrix} \arrow[ur] & \\
& & \begin{smallmatrix} a_1 \\ b_0 \\ a_0 \end{smallmatrix} \arrow[ur] & & & &
\end{tikzcd}$$

Clearly, the simple module a_1 belongs to Σ_1 but its injective envelope $\begin{smallmatrix} b_1 \\ a_1 \end{smallmatrix}$ is not projective. The above proof fails here because the module $\begin{smallmatrix} b_0 \\ a_0 \end{smallmatrix}$ is a projective-injective A-module and $\Omega^{-1}_{A^{(m)}} \begin{smallmatrix} b_0 \\ a_0 \end{smallmatrix} = a_1$ is not in $\operatorname{ind} A_{0,1}$.

Proposition 25 *Let A be representation-infinite and let T be a faithful exceptional $A^{(m)}$-module such that $\operatorname{pd} T \leq m$. Then there exists an exact sequence*

$$0 \longrightarrow A \xrightarrow{f_0} T_0 \xrightarrow{f_1} T_1 \xrightarrow{f_2} \cdots \xrightarrow{f_{m-1}} T_{m-1} \xrightarrow{f_m} L_m \longrightarrow 0$$

in $\operatorname{mod} A^{(m)}$ *such that*

(1) $T_i \in \operatorname{add} T$ *for all* i,
(2) $L_i = \operatorname{Coker} f_{i-1}$ *lies in* $\operatorname{add} \mathcal{L}_i(A^{(m)})$, *and*
(3) each of the induced monomorphisms $L_i \hookrightarrow T_i$ *is a minimal left* $\operatorname{add} T$-*approximation.*

PROOF. We construct this sequence by induction on $s < m$. First, since T is faithful and $\operatorname{pd} A_{A^{(m)}} = 0$, it follows from Lemma 23 that a minimal left $\operatorname{add} T$-approximation of $A_{A^{(m)}}$ is a monomorphism, and thus we have a short exact sequence

$$0 \longrightarrow A \xrightarrow{f_0} T_0 \xrightarrow{g_1} L_1 \longrightarrow 0$$

in $\operatorname{mod} A^{(m)}$.

Assume now that $s < m$ and that we have an exact sequence

$$0 \longrightarrow A \xrightarrow{f_0} T_0 \xrightarrow{f_1} T_1 \longrightarrow \cdots \xrightarrow{f_s} T_s$$

such that $T_i \in \operatorname{add} T,\ L_i = \operatorname{Coker} f_{i-1} \in \operatorname{add} \mathcal{L}_i(A^{(m)})$ and each of the induced monomorphisms $f_i' : L_i \hookrightarrow T_i$ is a minimal left $\operatorname{add} T$-approximation. Let $L_{s+1} = \operatorname{Coker} f_s$. If $L_{s+1} = 0$, then our sequence stops and it is of the form

$$0 \longrightarrow A \longrightarrow T_0 \longrightarrow T_1 \longrightarrow \cdots \longrightarrow T_s \longrightarrow 0$$

and all $L_i = 0$, $T_i = 0$ if $s < i \leq m$. This sequence clearly satisfies conditions (1)-(3). If $L_{s+1} \neq 0$, consider the short exact sequence

$$0 \longrightarrow L_s \xrightarrow{f_s'} T_s \xrightarrow{g_{s+1}} L_{s+1} \longrightarrow 0$$

in $\operatorname{mod} A^{(m)}$. By the induction hypothesis, $L_s \in \mathcal{L}_s(A^{(m)}) \subset \mathcal{L}_m(A^{(m)})$, therefore, by Lemma 23, L_s has a projective-injective injective envelope I, so that we have a commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & L_s \arrow[r, "f_s'"] \arrow[d, "\mathrm{id}"'] & T_s \arrow[r, "g_{s+1}"] \arrow[d] & L_{s+1} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & L_s \arrow[r] & I \arrow[r] & \Omega^{-1}_{A^{(m)}} L_s \arrow[r] & 0.
\end{tikzcd}$$

Since $L_s \in \mathcal{L}_s(A^{(m)})$ implies $L_s \leq \Sigma_s$, we have

$$L_{s+1} \leq \Omega^{-1}_{A^{(m)}} L_s \leq \Omega^{-1}_{A^{(m)}} \Sigma_s = \Sigma_{s+1} \leq \Sigma_m$$

(because $s+1 \leq m$). In particular $L_{s+1} \in \mathcal{L}_{s+1}(A^{(m)})$. By Lemma 23, a minimal left $\operatorname{add} T$-approximation $f'_{s+1} : L_{s+1} \to T_{s+1}$ is a monomorphism and in particular $T_{s+1} \neq 0$. We obtain the exact sequence

$$0 \longrightarrow A \xrightarrow{f_0} T_0 \xrightarrow{f_1} T_1 \longrightarrow \cdots \xrightarrow{f_s} T_s \xrightarrow{f_{s+1}} T_{s+1} \longrightarrow L_{s+1} \longrightarrow 0$$

where $f_{s+1} = f'_{s+1} \circ g_{s+1}$, and this completes the proof. $\square$

Theorem 26 *Let A be a hereditary algebra, $m \geq 1$ and $A^{(m)}$ be the m-th replicated algebra of A. If T is a faithful exceptional $A^{(m)}$-module with $\operatorname{pd} T \leq m$, then there exists an $A^{(m)}$-module X such that $T \oplus X$ is a tilting $A^{(m)}$-module and $\operatorname{pd} X \leq m$.*

PROOF. We may, by [23], assume that $A^{(m)}$, or equivalently, A, is representation-infinite. Then, by Proposition 25, there exists an exact sequence

$$0 \longrightarrow A \xrightarrow{f_0} T_0 \xrightarrow{f_1} T_1 \longrightarrow \cdots \xrightarrow{f_{m-1}} T_{m-1} \xrightarrow{f_m} L_m \longrightarrow 0$$

in $\operatorname{mod} A^{(m)}$ such that

(1) $T_i \in \operatorname{add} T$ for all i,
(2) $L_i = \operatorname{Coker} f_{i-1}$ lies in $\operatorname{add} \mathcal{L}_i(A^{(m)})$ (or, equivalently, $\operatorname{pd} L_i \leq i$), and
(3) each of the induced monomorphisms $L_i \hookrightarrow T_i$ is a minimal left $\operatorname{add} T$-approximation.

Actually, we have one of two cases. If there exists $p \leq m$ such that $L_p = 0$, then the above sequence reduces to

$$0 \longrightarrow A \xrightarrow{f_0} T_0 \xrightarrow{f_1} T_1 \longrightarrow \cdots \xrightarrow{f_p} T_p \longrightarrow 0$$

(with $T_i \in \operatorname{add} T$ for all i) and then T is clearly a tilting module. If not we may assume $L_m \neq 0$. We then prove by induction on s, with $1 \leq s \leq m$, that for all $i \geq 1$

$$\operatorname{Ext}^i_{A^{(m)}}(L_s, T) = 0. \tag{2}$$
$$\operatorname{Ext}^i_{A^{(m)}}(T, L_s) \cong \operatorname{Ext}^{i+s}_{A^{(m)}}(T, A). \tag{3}$$
$$\operatorname{Ext}^i_{A^{(m)}}(L_s, L_s) \cong \operatorname{Ext}^{i+s}_{A^{(m)}}(L_s, A). \tag{4}$$

Assume first that $s = 1$, and apply $\operatorname{Hom}_{A^{(m)}}(-, T)$ to the short exact sequence

$$0 \longrightarrow A \xrightarrow{f_0} T_0 \longrightarrow L_1 \longrightarrow 0. \tag{5}$$

This yields an exact sequence

$$\begin{aligned}\operatorname{Hom}_{A^{(m)}}(T_0,T) &\xrightarrow{\operatorname{Hom}_{A^{(m)}}(f_0,T)} \operatorname{Hom}_{A^{(m)}}(A,T) \longrightarrow \operatorname{Ext}^1_{A^{(m)}}(L_1,T)\\ &\longrightarrow \quad \operatorname{Ext}^1_{A^{(m)}}(T_0,T) \;=\; 0\end{aligned}$$

where the last equality follows from the exceptionality of T. Since f_0 is a left $\operatorname{add} T$-approximation, $\operatorname{Hom}_{A^{(m)}}(f_0,T)$ is surjective. Hence $\operatorname{Ext}^1_{A^{(m)}}(L_1,T)=0$. This implies (2), because $\operatorname{pd} L_1 \le 1$.

Applying now $\operatorname{Hom}_{A^{(m)}}(T,-)$ to (5) yields the exact sequence

$$0=\operatorname{Ext}^i_{A^{(m)}}(T,T_0)\to\operatorname{Ext}^i_{A^{(m)}}(T,L_1)\to\operatorname{Ext}^{i+1}_{A^{(m)}}(T,A)\to\operatorname{Ext}^{i+1}_{A^{(m)}}(T,T_0)=0,$$

hence $\operatorname{Ext}^i_{A^{(m)}}(T,L_1)\cong\operatorname{Ext}^{i+1}_{A^{(m)}}(T,A)$. This is just (3).

Finally, applying $\operatorname{Hom}_{A^{(m)}}(L_1,-)$ to (5) yields the exact sequence

$$\begin{aligned}0=\operatorname{Ext}^1_{A^{(m)}}(L_1,T_0) &\longrightarrow \operatorname{Ext}^1_{A^{(m)}}(L_1,L_1)\longrightarrow \operatorname{Ext}^2_{A^{(m)}}(L_1,A)\\ &\longrightarrow \operatorname{Ext}^2_{A^{(m)}}(L_1,T_0) \;=\; 0\end{aligned}$$

hence $\operatorname{Ext}^1_{A^{(m)}}(L_1,L_1)\cong\operatorname{Ext}^2_{A^{(m)}}(L_1,A)\,(=0)$, and thus (4), because $\operatorname{pd} L_1 \le 1$.

Let now $s>1$ and consider the short exact sequence

$$0\longrightarrow L_{s-1}\xrightarrow{f'_{s-1}} T_{s-1}\longrightarrow L_s\longrightarrow 0. \tag{6}$$

Applying $\operatorname{Hom}_{A^{(m)}}(-,T)$ yields an exact sequence

$$\begin{aligned}\operatorname{Hom}_{A^{(m)}}(T_{s-1},T) &\xrightarrow{\operatorname{Hom}_{A^{(m)}}(f'_{s-1},T)} \operatorname{Hom}_{A^{(m)}}(L_{s-1},T)\longrightarrow \operatorname{Ext}^1_{A^{(m)}}(L_s,T)\\ &\longrightarrow \quad \operatorname{Ext}^1_{A^{(m)}}(T_{s-1},T) \;=\; 0\end{aligned}$$

Since f'_{s-1} is a left $\operatorname{add} T$-approximation, we get $\operatorname{Ext}^1_{A^{(m)}}(L_s,T)=0$. The same long exact sequence yields, for $i\ge 2$,

$$\operatorname{Ext}^i_{A^{(m)}}(L_s,T)\cong\operatorname{Ext}^{i-1}_{A^{(m)}}(L_{s-1},T)=0$$

where the second equality follows from the induction hypothesis. This shows (2).

Applying $\operatorname{Hom}_{A^{(m)}}(T,-)$ to (6) yields the exact sequences

$$\begin{aligned}0=\operatorname{Ext}^i_{A^{(m)}}(T,T_{s-1}) &\longrightarrow \operatorname{Ext}^i_{A^{(m)}}(T,L_s)\longrightarrow \operatorname{Ext}^{i+1}_{A^{(m)}}(T,L_{s-1})\\ &\longrightarrow \operatorname{Ext}^{i+1}_{A^{(m)}}(T,T_{s-1}) \;=\; 0\end{aligned}$$

thus $\mathrm{Ext}^i_{A^{(m)}}(T, L_s) \cong \mathrm{Ext}^{i+1}_{A^{(m)}}(T, L_{s-1})$ and $\mathrm{Ext}^{i+1}_{A^{(m)}}(T, L_{s-1}) \cong \mathrm{Ext}^{i+s}_{A^{(m)}}(T, A)$, by the induction hypothesis. This shows (3).

Finally, applying $\mathrm{Hom}_{A^{(m)}}(L_s, -)$ to (6) yields the exact sequences

$$\begin{aligned} 0 = \mathrm{Ext}^i_{A^{(m)}}(L_s, T_{s-1}) \longrightarrow\ & \mathrm{Ext}^i_{A^{(m)}}(L_s, L_s) \longrightarrow \mathrm{Ext}^{i+1}_{A^{(m)}}(L_s, L_{s-1}) \\ & \longrightarrow \mathrm{Ext}^{i+1}_{A^{(m)}}(L_s, T_{s-1}) = 0 \end{aligned}$$

hence $\mathrm{Ext}^i_{A^{(m)}}(L_s, L_s) \cong \mathrm{Ext}^{i+1}_{A^{(m)}}(L_s, L_{s-1})$. Similarly, applying $\mathrm{Hom}_{A^{(m)}}(L_s, -)$ to the short exact sequence

$$0 \longrightarrow L_{s-2} \xrightarrow{f'_{s-2}} T_{s-2} \longrightarrow L_{s-1} \longrightarrow 0.$$

yields $\mathrm{Ext}^{i+1}_{A^{(m)}}(L_s, L_{s-1}) \cong \mathrm{Ext}^{i+2}_{A^{(m)}}(L_s, L_{s-2})$. Continuing in this way, one gets eventually $\mathrm{Ext}^i_{A^{(m)}}(L_s, L_s) \cong \mathrm{Ext}^{i+s}_{A^{(m)}}(L_s, L_0)$ with $L_0 = A$. This shows (4) and completes the proof of our claim.

Let now $s = m$. We deduce that, for all $i \geq 1$

$$\begin{aligned} &\mathrm{Ext}^i_{A^{(m)}}(L_m, T) = 0. \\ &\mathrm{Ext}^i_{A^{(m)}}(T, L_m) \cong \mathrm{Ext}^{m+i}_{A^{(m)}}(T, A) = 0 \qquad \text{since } \mathrm{pd}\, T \leq m. \\ &\mathrm{Ext}^i_{A^{(m)}}(L_m, L_m) \cong \mathrm{Ext}^{m+i}_{A^{(m)}}(L_m, A) = 0 \text{ since } \mathrm{pd}\, L_m \leq m. \end{aligned}$$

This shows that $T \oplus L_m$ is a tilting $A^{(m)}$-module, and completes the proof of the theorem. □

Remark 27 *Observe that T has usually many possible complements. Our proof constructs only one.*

Corollary 28 *Let A be a hereditary algebra, $m \geq 1$ and $A^{(m)}$ be the m-replicated algebra of A. Let T be a basic, faithful, exceptional $A^{(m)}$-module with $\mathrm{pd}\, T \leq m$ and such that the number of indecomposable summands of T is equal to the rank of the Grothendieck group of $A^{(m)}$. Then T is a tilting $A^{(m)}$-module.*

5 Tilting modules and tilting objects

5.1 Main result

Let A be a hereditary algebra, $m \geq 1$ and X be an object in the m-cluster category $\mathcal{C}_m(A)$. Then X is said to be *basic* (or *multiplicity free*) if, whenever

$X = \oplus_{i=1}^{t} X_i$ where all the X_i are indecomposable, we have $X_i \neq X_j$ for $i \neq j$. The object X is called *exceptional* if $\mathrm{Ext}^i_{\mathcal{C}_m(A)}(X,X) = 0$ for all i with $1 \leq i \leq m$ and it is called *tilting* if it is exceptional and the number of isomorphism classes of its indecomposable summands is equal to the rank of the Grothendieck group of A (see [25]).

Let now T be an exceptional $A^{(m)}$-module. Then we can always write T in the form $T = T' \oplus P$, where P is projective-injective and T' has no projective-injective indecomposable summands. We say that T is an *$\mathcal{L}_m$-exceptional module* if $T' \in \operatorname{add} \mathcal{L}_m(A^{(m)})$ (or, equivalently, if $\operatorname{pd} T \leq m$).

In this section, we always assume our exceptional objects and modules to be basic. By abuse of notation, modules will often be denoted by the same letter even when considered as objects in different categories. We now prove Theorem 3.

Theorem 29 *Let A be a hereditary algebra and $A^{(m)}$ be its m-th replicated algebra. There is a one-to-one correspondence:*

$$\{\textit{basic } \mathcal{L}_m - \textit{exceptional modules}\} \leftrightarrow \{\textit{basic exceptional objects in } \mathcal{C}_m(A)\},$$

which is given by $T = T' \oplus P \mapsto \pi(T')$.

PROOF. Let $T = T' \oplus P$ be a basic $\mathcal{L}_m$-exceptional module. We claim that $X = \pi(T')$ is an exceptional object in $\mathcal{C}_m(A)$, that is, $\mathrm{Ext}^i_{\mathcal{C}_m(A)}(X,X) = 0$ for $1 \leq i \leq m$. By the definition of $\mathrm{Ext}^i_{\mathcal{C}_m(A)}$ this amounts to proving that

$$\mathrm{Hom}_{D^b(\mathrm{mod} A)}(X_x, \tau^{-s} X_y[ms+i]) = 0 \tag{7}$$

(where $\tau = \tau_{\mathcal{D}^b(\mathrm{mod}\, A)}$) for all $s \in \mathbb{Z}$, all i such that $1 \leq i \leq m$ and all indecomposable summands X_x, X_y of X. Denote by T_x and T_y the indecomposable $A^{(m)}$-modules lying in $\mathcal{L}_m(A^{(m)})$ which correspond to X_x and X_y, respectively. We show equation (7) by distinguishing various cases according to the value of s.

(1) If $s = 0$ then the equation (7) holds for all i since T is an $\mathcal{L}_m-$exceptional module.
(2) If $s = -1$ then we have for all i

$$\begin{aligned}
\mathrm{Hom}_{D^b(\mathrm{mod} A)}(X_x, \tau X_y[-m+i]) &\cong \underline{\mathrm{Hom}}_{\hat{A}}(T_x, \tau_{\hat{A}}\, \Omega^{m-i}_{\hat{A}}\, T_y) && (8)\\
&\cong D\mathrm{Ext}^1_{\hat{A}}(\Omega^{m-i}_{\hat{A}}\, T_y, T_x) && (9)\\
&\cong D\mathrm{Ext}^{m+1-i}_{\hat{A}}(T_y, T_x) && (10)\\
&\cong D\mathrm{Ext}^{m+1-i}_{A^{(m)}}(T_y, T_x) && (11)\\
&= 0 && (12)
\end{aligned}$$

where (8) follows from Theorem 4, (9) is the Auslander-Reiten formula in $\mathcal{D}^b(\operatorname{mod} A)$, (10) follows from the definition of Ω, (11) holds because $T \in \operatorname{add} \mathcal{L}_m(A^{(m)})$ and (12) because T is exceptional.

(3) If $s \leq -2$ then $ms + i \leq -2m + i \leq -m$, so $X_y[ms + i]$ lies in some $(\operatorname{ind} A)[j]$ with $j < 0$ except in the case where $s = -2,\ i = m$ and $X_y \in (\operatorname{ind} A)[m]$, in which case $X_y[ms + i] \in (\operatorname{ind} A)[0]$, and then $\tau^{-s} X_y[ms + i](= \tau^2 X_y[ms+i])$ lies in some $(\operatorname{ind} A)[j]$ with $j < 0$. In either case, there are no non-zero morphisms from X_x to $\tau^{-s} X_y[ms + i]$.

(4) If $s \geq 1$, then $ms + i \geq m + i \geq m$ so the only possibility to have a non-zero morphism from X_x to $\tau^{-s} X_y[ms + i]$ is when $s = 1, i = 1, X_x = P[m]$ for some indecomposable projective A-module P, and $\tau^{-1} X_y \in (\operatorname{ind} A)[0]$. But then $\tau^{-1} X_y[ms + i] \in (\operatorname{ind} A)[m + 1]$ and there is no non-zero morphism from $P[m]$ to $(\operatorname{ind} A)[m + 1]$.

Conversely, assume that $X = \pi(T')$ is a basic exceptional object in $\mathcal{C}_m(A)$. We claim that T' is a basic $\mathcal{L}_m-$exceptional $A^{(m)}$-module. Clearly, T' is basic and of finite projective dimension. Moreover $T' \in \mathcal{L}_m(A^{(m)})$, by Theorem 20. Suppose that there exist indecomposable summands T_x, T_y of T' such that $\operatorname{Ext}^i_{A^{(m)}}(T_x, T_y) \neq 0$ for some i with $1 \leq i \leq m$. Then

$$\begin{aligned}\underline{\operatorname{Hom}}_{\hat{A}}(T_y, \tau_{\hat{A}} \Omega^{i-1}_{\hat{A}} T_x) &\cong D\operatorname{Ext}^1_{\hat{A}}(\Omega^{i-1}_{\hat{A}} T_x, T_y) \cong D\operatorname{Ext}^i_{\hat{A}}(T_x, T_y)\\ &\cong D\operatorname{Ext}^i_{A^{(m)}}(T_x, T_y)\end{aligned}$$

implies that

$$\operatorname{Hom}_{\mathcal{D}^b(\operatorname{mod} A)}(X_y, \tau X_x[-i + 1]) \neq 0$$

(where X_x, X_y denote, as before, the indecomposable summands of X which correspond to T_x, T_y, respectively). Thus we have

$$\begin{aligned}\operatorname{Ext}^i_{\mathcal{C}_m(A)}(X_x, X_y) &\cong \operatorname{Ext}^1_{\mathcal{C}_m(A)}(X_x[-i + 1], X_y)\\ &\cong \operatorname{Hom}_{\mathcal{C}_m(A)}(X_y, \tau_{\mathcal{C}_m(A)} X_x[-i + 1])\\ &\neq 0\end{aligned}$$

contradicting the hypothesis that X is an exceptional object in $\mathcal{C}_m(A)$ $\square$

Corollary 30 *Let A be a hereditary algebra and $A^{(m)}$ be its m-th replicated algebra. There is a one-to-one correspondence*

$$\{\textit{basic } \mathcal{L}_m - \textit{tilting modules}\} \leftrightarrow \{\textit{basic tilting objects in } \mathcal{C}_m A\},$$

which is given by $T = T' \oplus P \mapsto \pi(T')$.

PROOF. Assume T is a basic $\mathcal{L}_m$-tilting module, then $T = T' \oplus P$ where P has nm indecomposable summands (here n is the rank of the Grothendieck

group of A) while T' has n indecomposable summands. But then $X = \pi(T')$ has also n indecomposable summands. Since, by Theorem 29, X is exceptional, it is tilting.

Conversely, if X is a tilting object in $\mathcal{C}_m(A)$ then it has n indecomposable summands. Let $T = T' \oplus P$ where P is the direct sum of all projective-injective indecomposable $A^{(m)}$-modules and T' is such that $\pi(T') = X$. Since P has nm indecomposable summands, and T' has n, then T has $nm+n$ indecomposable summands. But $nm+n$ is equal to the rank of the Grothendieck group of $A^{(m)}$ and, by Corollary 28, T is a tilting module. $\square$

5.2 Application to the case $m = 1$

In [3], it is shown that there is a bijection between the tilting objects in the cluster category $\mathcal{C}(A) = \mathcal{C}_1(A)$ and the $\mathcal{L}$-tilting modules in the duplicated algebra $\overline{A} = A^{(1)}$. Now, recall that in this context, tilting modules are of projective dimension 1. By Corollary 18, this is equivalent to the fact that the non projective-injective indecomposable summands lie in $\mathcal{L}_{\overline{A}} = \mathcal{L}_1(\overline{A})$. Therefore, we have the following corollary.

Corollary 31 *Let A be a hereditary algebra. There is a one-to-one correspondence*

$$\{\textit{basic tilting } \overline{A}\textit{-modules}\} \leftrightarrow \{\textit{basic cluster tilting objects in } \mathcal{C}(A)\},$$

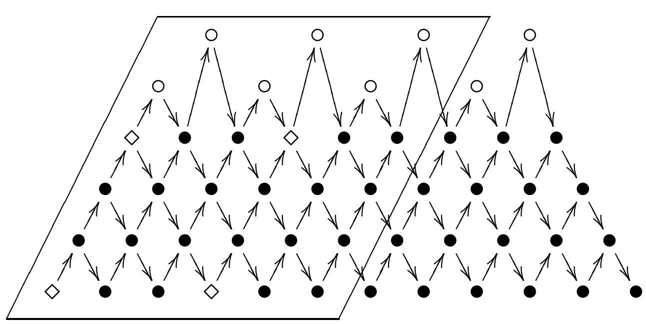
which is given by $T = T' \oplus P \mapsto \pi(T')$.

5.3 Example

Let A be given by the quiver

$$1 \longleftarrow 2 \longleftarrow 3 \longleftarrow 4$$

Then the Auslander-Reiten quiver of $A^{(2)}$ is given by



where we have indicated the 2-left part $\mathcal{L}_2(A^{(2)})$. We have also indicated an $\mathcal{L}_2$-tilting module $T = T' \oplus P$, where $T' \in \operatorname{add} \mathcal{L}_2$. The summands of T' are indicated by diamonds and the (projective-injective) summands of P by circles.

References

[1] I. Assem, On representations of the bimodule DA, *Tsukuba J. Math.* **9**, No.2 (1985) 217-232.

[2] I. Assem, A. Beligiannis and N. Marmaridis, Right triangulated categories with right semi-equivalences, *CMS Conference Proceedings* **24**, (1998) 17-37.

[3] I. Assem, T. Brüstle, R. Schiffler and G. Todorov, Cluster categories and duplicated algebras. to appear in *J. Algebra*.

[4] I. Assem and Y. Iwanaga, On a class of representation-finite QF-3 algebras, *Tsukuba J. Math.* **11**, (1987) 199-210.

[5] I. Assem, J. Nehring and W. Schewe, Fundamental domains and duplicated algebras, *CMS Conference Proceedings* **11**, (1991) 25-51.

[6] M. Auslander and I. Reiten, Applications of contravariantly finite subcategories, *Advances in Math.* **86**, no. 1 (1991) 111-152.

[7] M. Auslander, I. Reiten and S.O. Smalø, *Representation Theory of Artin Algebras* Cambridge Studies in Advanced Math. 36, (Cambridge University Press, 1995).

[8] K. Baur and R. Marsh, A geometric description of m-cluster categories, preprint (2006).

[9] K. Bongartz, Tilted algebras, *Representations of algebras (Puebla, 1980)*, 26-38, Lecture Notes in Math., 903, Springer, Berlin-New York, 1981.

[10] K. Bongartz and P. Gabriel, Covering spaces in representation-theory, *Invent. Math.* **65** (1981/82), no. 3, 331–378.

[11] A. Buan, R. Marsh, M. Reineke, I. Reiten and G. Todorov, Tilting theory and cluster combinatorics, *Adv. Math.* **204** (2006), 572-612.

[12] P. Caldero, F. Chapoton and R. Schiffler, Quivers with relations arising from clusters (A_n case), *Trans. Amer. Math. Soc.* **358** (2006), no.3, 1347-1364.

[13] F. U. Coelho, D. Happel and L. Unger, Complements to partial tilting modules, *J. Algebra* **170** (1994), no. 1, 184-205.

[14] S. Fomin and N. Reading, Generalized cluster complexes and Coxeter combinatorics, preprint (2005)

[15] S. Fomin and A. Zelevinsky, Cluster algebras I. Foundations, *J. Amer. Math. Soc.* **15**(2), (2002), 497-529 (electronic)

[16] S. Fomin and A. Zelevinsky, Cluster algebras II. Finite type classification, *Inventiones Mathematicae* **154**(1), (2003), 63-121.

[17] D. Happel, *Triangulated Categories in the Representation Theory of Finite Dimensional Algebras* London Mathematical Society. Lecture Notes Series 119, (Cambridge University Press, 1988).

[18] D. Happel, On the derived category of a finite dimensional algebra, *Comment. Math. Helv.* **62** (1987), no. 3, 339-389.

[19] D. Happel, I. Reiten and S.O. Smalø, Tilting in abelian categories and quasi tilted algebras, *Memoirs Amer. Math. Soc.* **575**, (1996).

[20] D. Hughes and J. Waschbüsch, Trivial extensions of tilted algebras *Proc. London Math. Soc.* **46** (1983) 347-364.

[21] B. Keller, On triangulated orbit categories, *Doc. Math.* **10** (2005), 551-581 (electronic).

[22] B. Keller and I. Reiten, Cluster-tilted algebras are Gorenstein and stably Calabi-Yau, preprint (2005).

[23] J. Rickard and A. Schofield, Cocovers and tilting modules, *Math. Proc. Cambridge Philos. Soc.* **106** (1989), no. 1, 1-5.

[24] C.M. Ringel, *Tame algebras and integral quadratic forms,* Lecture Notes in Math. 1099, (Springer Verlag, 1984).

[25] H. Thomas, Defining an m-cluster category, preprint (2005).

[26] H. Tachikawa, Representations of trivial extensions of hereditary algebras, *Proc. ICRA II* , Lecture Notes in Math. 832, (Springer Verlag 1980) 579-599.

[27] K. Yamagata, Extensions over hereditary artinian rings with self-dualities I. *J. Algebra* **73** (2) (1981) 386-433.

[28] B. Zhu, Generalized cluster complexes via quiver representations, preprint (2006).